

# Scalar lattice gauge theory

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Scalar lattice gauge theories are models for scalar fields with local symmetries. No fundamental gauge fields, or link variables in a lattice regularization, are introduced. The latter rather arise as collective excitations composed from scalars. We argue that a suitable region in the rich phase diagram of scalar lattice gauge theory belongs to the same universality class as usual lattice gauge theories formulated in terms of links. This therefore leads to the same predictions in the continuum limit, in particular confinement. We also present an equivalent a formulation using only gauge invariant composite scalar fields. We speculate about the use of linear link variables for the description of the confinement regime in non-abelian gauge theories. Scalar lattice gauge theories or their fermionic counterpart may also be helpful for a realization of gauge theories in ultracold atoms.

## I. INTRODUCTION

Can gauge bosons be composites of other fundamental degrees of freedom? In the high momentum regime of asymptotically free non-abelian gauge theories the gauge bosons are effectively massless particles. If an underlying theory with different degrees of freedom, as scalars or fermions, is to produce massless collective states there needs to be a reason for this. The obvious reason for collective massless spin-one bosons is a local gauge symmetry. In the perturbative regime this forbids a mass term for the gauge bosons. We therefore investigate models with a local gauge symmetry, but without introducing gauge bosons as fundamental degrees of freedom.

A suitable setting for regularized gauge theories are lattice theories. We will follow this road and formulate our models on an euclidean space-time lattice. In contrast to Wilson's original proposal [1] we will not use link variables as basic degrees of freedom. We rather want to know if it is possible to formulate gauge theories in terms of fundamental fermions or scalars  $\psi(x)$  which are associated to each lattice point  $x$ , and transform under local gauge transformations as  $\psi(x) \rightarrow V(x)\psi(x)$ ,  $V^\dagger V = 1$ . For the case of fermions the presence of local gauge symmetries has already been observed in a discussion of lattice spinor gravity [2]. Furthermore, any higher-dimensional model which exhibits diffeomorphism symmetry and a collective metric or vielbein can yield four-dimensional gauge theories [3]. They arise after dimensional reduction for an arbitrary “internal” space with isometries [4–6].

While the formulation of an action with local gauge symmetry is rather straightforward, it is not clear if generalized gauge theories based on scalars or fermions will produce the striking dynamical features as asymptotic freedom and confinement that characterize standard non-abelian gauge theories. We argue that this is indeed possible for a suitable region in the space of parameters. Gauge theories for “fundamental” scalars or fermions may thus offer interesting perspectives for a realization of  $d$ -dimensional gauge theories by ultracold atoms [7–9].

In this paper we concentrate on “fundamental” scalar fields rather than on fermions. The reason for this is an

easy access to such theories by numerical simulations. Such simulations will be very useful to verify the suggestions made in the present work. We formulate lattice models purely in terms of scalar fields and want to understand how collective link variables and associated gauge fields arise dynamically. The generalization to “fundamental” fermions is briefly discussed in the conclusions.

More in detail, we will investigate models with  $MN$  complex scalar fields  $\chi_i^a(x)$ , with  $i = 1 \dots N$  the “gauge index” on which  $SU(N) \times U(1)$  gauge transformations act as  $\chi_i^a(x) \rightarrow V_{ij}(x)\chi_j^a(x)$ . We consider  $M$  flavors, labeled by the index  $a$ . Gauge invariant combinations of scalars can be formed as  $M^{ab}(x) = (\chi_i^a(x))^* \chi_i^b(x)$ . A functional integral for which the action can be written uniquely in terms of the hermitean  $M \times M$  matrices  $M(x)$  is invariant under local gauge transformations without the need to introduce explicit link variables.

An example for such an action has the naive continuum limit

$$S = \int d^4x [\bar{m}^2 \text{tr} M^2 + \bar{\lambda} \text{tr} \{ M^2 \partial_\mu M \partial_\mu M \} - \frac{3}{2} \text{tr} \{ M \partial_\mu M M \partial_\mu M \}]. \quad (1)$$

This action is positive for  $\bar{m}^2 > 0$ ,  $\bar{\lambda} > 3/2$ , with minimum for  $M(x) = 0$ . It is characterized by a derivative interaction involving four powers of  $M(x)$  or eight powers of  $\chi(x)$ . Besides local  $SU(N) \times U(1)$  gauge symmetry and space-time symmetries it is also invariant under global  $SU(M)$ -flavor transformations acting on the flavor indices of  $\chi$  or  $M$ . A precise lattice formulation of the action will be given in the next section and is important for the detailed understanding. Extensions of this action will be discussed in later parts of this paper. (A kinetic term  $\sim \text{tr} \{ \partial_\mu M \partial_\mu M \}$  could be added without changing the qualitative features.)

Due to the complicated structure of the interaction it is not easy to guess the dynamics of the low momentum excitations of this model. We will argue that for suitable  $\bar{m}^2$  and  $\bar{\lambda}$  it actually belongs to the same universality class as a pure  $SU(N) \times U(1)$  gauge theory. For this purpose we will use collective link variables and perform a Hubbard-Stratonovich-type transformation [10, 11] of the functional

integral. On the other side, we will show that the functional integral can be reformulated in terms of  $M(x)$  instead of  $\chi(x)$ . This eliminates any direct appearance of the local gauge transformations since  $M(x)$  is invariant. The formulation in terms of  $M$  adds an effective non-polynomial potential term to the action (1) such that its minimum occurs for  $\langle M \rangle \neq 0$ .

One and the same theory is therefore formulated in terms of different degrees of freedom: colored scalars  $\chi$ , invariant scalars  $M$ , or link variables coupled to colored scalars. Understanding the connections between the equivalent formulations may be useful for the understanding of the structure of gauge theories, in particular if fermions (quarks) are added. We will speculate that a coupled system of gauge bosons and scalars may provide for a comparatively simple description of the confinement regime.

This paper is organized as follows: In sect. II we discuss the formulation of a lattice theory with local gauge symmetry in terms of colored scalars  $\chi$ . We introduce scalar bilinears that transform as link variables and partly reformulate the action in terms of those link bilinears. In contrast to the usual link variables in lattice gauge theories the link bilinears are not constrained to be unitary matrices - they can take arbitrary values. For this reason we discuss next in sect. III a formulation of lattice gauge theories in terms of unconstrained link variables - linear lattice gauge theory. For a suitable region in parameter space linear lattice gauge theory belongs to the same universality class as the standard “non-linear lattice gauge theories”. The additional massive degrees of freedom play no role in the long-distance or continuum limit. In this section we also interpret the increase of the running non-abelian gauge coupling at large distances as the decrease of the expectation value of a scalar field. The confinement or strong coupling regime is simply identified with a zero expectation of this scalar field.

In sect. IV we resume the discussion of the scalar lattice gauge theory introduced in sect. II. We employ a formalism with collective link variables and perform a transformation of the functional integral to a formulation with linear link variables and colored scalar fields. Even though the interactions between the scalars and gauge bosons (link variables) are rather complicated, the basic structure of the combined action for link variables and scalars suggests that the model is in the same universality class as linear lattice gauge theory for a suitable region in parameter space. It is then also in the same universality class as standard lattice gauge theory. In sect. V we propose explicitly a choice of parameters for which we expect that the model realizes a weak coupling lattice gauge theory, with confinement scale far below the lattice cutoff. In particular, we argue that a suitably taken limit  $\bar{\lambda} \rightarrow \infty$  is equivalent to standard lattice gauge theory.

In sect. VI we turn to a pure singlet formulation of scalar lattice gauge theory in terms of the “composite” scalar fields  $M(x)$ . This realizes a formulation of gauge theories only in terms of gauge invariant quantities. Our conclusions are drawn in sect. VII.

## II. INVARIANT LATTICE ACTION AND COLLECTIVE LINK VARIABLES

We start with  $N \times M$  dimensionless complex scalar fields  $\chi_i^a(x)$ . Here  $i = 1 \dots N$  is a color index, and  $a = 1 \dots M$  a flavor index. The coordinates  $x^\mu$  denote points of a  $d$ -dimensional hypercubic lattice,  $x^\mu = an^\mu$ , with integer  $n^\mu$  and  $a$  the lattice distance. Periodic or other boundary conditions may be imposed such that the number of lattice points  $\mathcal{N}$  is finite, with continuum limit  $\mathcal{N} \rightarrow \infty$  taken at the end. The local gauge symmetries act on the color index,  $\chi_i^a(x)' = V_{ij}(x)\chi_j^a(x)$ ,  $V^\dagger V = 1$ .

For a functional integral

$$Z = \int \mathcal{D}\chi e^{-S[\chi]} = \left( \prod_x \prod_i \prod_a \int d\chi_i^a(x) d\chi_i^a(x)^* \right) e^{-S[\chi]} \quad (2)$$

the dynamics of the model is determined by the form of the microscopic action  $S$ . We will concentrate on an action that can be written in terms of the scalar bilinears

$$M^{ab}(x) = (\chi_i^a(x))^* \chi_i^b(x), \quad (3)$$

as well as lattice derivatives thereof. Here we define lattice derivatives by

$$\partial_\mu f(x) = (f(x + e_\mu) - f(x))/a, \quad (4)$$

with  $e_\mu$  a lattice unit vector in the  $\mu$ -direction. We will represent  $M^{ab}(x)$  as an  $M \times M$  matrix  $M(x)$ . Eq. (3) involves a sum over the color indices  $i$ . Thus  $M(x)$  is invariant under local gauge transformations, guaranteeing the gauge invariance of the action. Generalizations and details of possible actions with local gauge symmetry are discussed in appendix A.

Our lattice regularization of the action (1) reads

$$S = \bar{S}_p + \bar{S}_l, \quad \bar{S}_p = \sum_{\text{plaquettes}} \mathcal{S}_p, \quad \bar{S}_l = \sum_{\text{links}} \mathcal{S}_l, \quad (5)$$

where the sum over links is  $\sum_x \sum_\mu$ , while the sum over plaquettes corresponds to  $\sum_x \sum_\nu \sum_{\mu < \nu}$ . The “link action”  $\mathcal{S}_l$  reads

$$\begin{aligned} \mathcal{S}_l = & \frac{1}{4} \left( \bar{\lambda} - \frac{d-1}{2} \right) \text{tr} \left\{ [M(x + e_\mu) + M(x)]^2 \right. \\ & \times [M(x + e_\mu) - M(x)]^2 \} \\ & + \frac{\bar{m}^2}{4d} a^{\frac{d+2}{2}} \text{tr} \{ [M(x + e_\mu) + M(x)]^2 \}. \end{aligned} \quad (6)$$

For  $\bar{\lambda} \geq (d-1)/2$  and  $\bar{m}^2 \geq 0$  it is positive definite.

The “plaquette action”  $\mathcal{S}_p$  involves the matrices  $M(x) = M_1, M(x+e_\mu) = M_2, M(x+e_\nu) = M_3$  and  $M(x+e_\mu+e_\nu) = M_4$  for the four points  $(x, x+e_\mu, x+e_\nu, x+e_\mu+e_\nu)$  belonging to a plaquette  $(x; \mu\nu)$ . It is defined as

$$\begin{aligned} \mathcal{S}_p = & \frac{1}{8} \text{tr} \{ (M_1^2 + M_4^2)(M_3 - M_2)^2 \\ & + (M_2^2 + M_3^2)(M_4 - M_1)^2 \\ & - 2M_1(M_3 - M_2)M_4(M_3 - M_2) \\ & - 2M_2(M_4 - M_1)M_3(M_4 - M_1) \}. \end{aligned} \quad (7)$$

In appendix A we show  $\mathcal{S}_p \geq 0$ . We also establish that the lattice action (5) has (for  $d = 4$ ) the naive continuum limit (1).

As a functional of arbitrary  $M(x)$  the action (5) would have a minimum for any configuration where  $M(x) = M_0$  for  $x$  even, and  $M(x) = -M_0$  for  $x$  odd, where for even  $x$  one has  $\sum_{\mu} n^{\mu} = \text{even}$ , while for odd  $x$  the sum of the  $d$  integers  $n^{\mu}$  is odd. Two neighboring sites belonging to a link  $(x; \mu)$  have then opposite  $M_0$  such that  $\mathcal{S}_l$  vanishes. Furthermore, one concludes from  $M_4 = M_1, M_3 = M_2$  that  $\mathcal{S}_p$  is zero as well. However,  $M$  is a composite (3) of the scalars  $\chi$  and cannot take arbitrary values. For example, the diagonal elements obey  $M^{aa} \geq 0$ . The only way to realize both matrices  $M_0$  and  $-M_0$  is for  $\chi_0 = 0, M_0 = 0$ . This defines the minimum of the action or the ground state. An understanding of the fluctuations is a rather complicated task due to the complex structure of the derivative terms. The issue will become more clear if we discuss  $\mathcal{S}_p$  in some more detail.

A central role of this work will be played by link bilinears. They involve scalars at neighboring sites, with flavor indices contracted,

$$\tilde{L}_{ij}(x; \mu) = \chi_i^a(x) (\chi_j^a(x + e_{\mu}))^*. \quad (8)$$

One may consider the link bilinears as complex  $N \times N$ -matrices  $\tilde{L}(x; \mu)$ . Under local unitary gauge transformations, represented by unitary  $N \times N$  matrices  $V(x)$ , the links transform as

$$\tilde{L}(x; \mu) \rightarrow V(x) \tilde{L}(x; \mu) V^{\dagger}(x + e_{\mu}), \quad (9)$$

similar to the links in lattice gauge theories.

It is instructive to investigate invariants constructed from link bilinears. A plaquette  $(x; \mu\nu)$  can be used for defining the invariant

$$\begin{aligned} \tilde{P}(x; \mu, \nu) = & \text{tr} \{ \tilde{L}(x; \mu) \tilde{L}(x + e_{\mu}; \nu) \\ & \times \tilde{L}^{\dagger}(x + e_{\nu}; \mu) \tilde{L}^{\dagger}(x; \nu) \}. \end{aligned} \quad (10)$$

Defining links with negative directions as

$$\tilde{L}(x + e_{\mu}; -\mu) = \tilde{L}^{\dagger}(x; \mu), \quad (11)$$

the invariant  $\tilde{P}$  corresponds to the trace of the product of four link matrices around the plaquette. The notation  $\tilde{P}(x; \mu, \nu)$  denotes a start at  $x$ , first link in direction  $\mu$ , second in direction  $\nu$ , third direction  $-\mu$ , and fourth direction  $-\nu$ . We observe that, in general,  $\tilde{P}$  has an orientation,

$$\tilde{P}(x; \nu, \mu) = \tilde{P}^*(x; \mu, \nu). \quad (12)$$

We will see below that the plaquette invariants  $\tilde{P}$  play an important role for the understanding of the plaquette action  $\mathcal{S}_p$  and for the close relation of our model to standard lattice gauge theories.

Using the definitions (3), (8), we can write  $\tilde{P}$  as a product of invariants,

$$\tilde{P}(x; \mu, \nu) = \text{Tr} \{ M(x) M(x + e_{\mu}) M(x + e_{\mu} + e_{\nu}) M(x + e_{\nu}) \}. \quad (13)$$

If we consider only one flavor we would have  $M(x) = M^*(x)$  such that  $\tilde{P}(x; \mu, \nu) = \tilde{P}^*(x; \mu, \nu)$ . Then  $\tilde{P}$  is blind to the angles in the decomposition  $\chi_i(x) = \tilde{U}_{i1}(x) k(x)$ ,  $k(x) \in \mathbb{R}$ ,  $\tilde{U}^{\dagger} \tilde{U} = 1$ , since  $M(x) = k^2(x)$ . Since such models cannot reproduce the angular dependencies of the plaquette invariant in standard lattice gauge theories we will not pursue them further in this note. We will discuss in the appendix B for which choice of the number of flavors  $M$  one may expect to find the same universality class as for a standard lattice gauge theory. Typically, this requires a minimal  $M$  for a given  $N$ . We will in the following use a notation for  $\chi$  as an  $N \times M$  matrix, where the first index is the color index and the second the flavor index,  $\chi_{ia}(x) \equiv \chi_i^a(x)$ . Correspondingly, the  $N \times N$  matrices  $\tilde{L}(x, \mu)$  and the  $M \times M$  matrices  $M(x)$  obey

$$\tilde{L}(x; \mu) = \chi(x) \chi^{\dagger}(x + e_{\mu}), \quad M(x) = \chi^{\dagger}(x) \chi(x). \quad (14)$$

We define a type of kinetic term for the link variables by

$$\mathcal{L}_p = \frac{1}{4} \text{tr} \{ \tilde{H}_1^{\dagger} \tilde{H}_1 + \tilde{H}_2^{\dagger} \tilde{H}_2 \}, \quad (15)$$

with

$$\begin{aligned} \tilde{H}_1 &= \tilde{L}(x; \mu) \tilde{L}(x + e_{\mu}; \nu) - \tilde{L}(x; \nu) \tilde{L}(x + e_{\nu}; \mu), \\ \tilde{H}_2 &= \tilde{L}(x + e_{\mu}; \nu) \tilde{L}^{\dagger}(x + e_{\nu}; \mu) - \tilde{L}^{\dagger}(x; \mu) \tilde{L}(x; \nu). \end{aligned} \quad (16)$$

This establishes  $\mathcal{L}_p \geq 0$ . We can write  $\mathcal{L}_p$  as a sum of invariants  $\sim \tilde{P}$  plus other terms, as explained in more detail in appendix A,

$$\begin{aligned} \mathcal{L}_p = & \frac{1}{4} \text{tr} \{ M_1 (M_3 - M_2) M_4 (M_3 - M_2) \\ & + M_2 (M_4 - M_1) M_3 (M_4 - M_1) \}. \end{aligned} \quad (17)$$

Comparing with eq. (13) the terms in eq. (17) with one matrix at each site can be identified with  $\tilde{P}$  or  $\tilde{P}^*$ . We observe that the expression (17) is invariant under lattice rotations by  $\pi/2$  and lattice reflections. We define for later purposes

$$\mathcal{S}_p = \sum_{\text{plaquettes}} \mathcal{L}_p(x; \mu, \nu). \quad (18)$$

which is different from  $\tilde{S}_p = \mathcal{A}_p - \mathcal{S}_p$  in eq. (5).

Indeed, we can write the plaquette action  $\mathcal{S}_p$  as a difference of two positive semidefinite terms

$$\mathcal{S}_p = \mathcal{A}_p - \mathcal{L}_p. \quad (19)$$

Here

$$\mathcal{A}_p = \frac{1}{8} \text{tr} \{ (M_1^2 + M_4^2) (M_3 - M_2)^2 + (M_2^2 + M_3^2) (M_4 - M_1)^2 \} \quad (20)$$

has a comparatively simple structure. It has the tendency to suppress inhomogeneities in the configuration  $M(x)$ . However, it competes with the plaquette term  $\sim \mathcal{L}_p$  which has a negative sign. The most difficult part in the understanding of the action (5) is due to the part  $-\mathcal{S}_p$  which favors nonzero values for the link bilinears. The point  $\chi = 0$

has zero weight in field space such that all relevant configurations will have  $\chi \neq 0$ . For  $\chi \neq 0$  the positive term  $\mathcal{A}_p$  and the negative term  $-\mathcal{L}_p$  tend to drive the fluctuations into opposite directions, somewhat similar to frustrated systems. It is this competition that will finally be responsible for the confinement characteristic for standard non-abelian gauge theories.

One may wonder if a simplification can be achieved by introducing explicit link variables via a Hubbard-Stratonovich type transformation of the functional integral. This will be done in sect. IV. However, the corresponding link variables will be arbitrary  $N \times N$  matrices, in contrast to the unitary matrices employed in standard lattice gauge theories. For this purpose we generalize in the next section the formulation of lattice gauge theories in terms of links to a setting with unconstrained link variables.

### III. LINEAR LATTICE GAUGE THEORY

In order to make contact with the conventional formulation of lattice gauge theories we first replace the link bilinears  $\tilde{L}(x; \mu)$  by link variables  $L(x; \mu)$ . (A more precise connection will be given in the next section.) The link variables  $L$  have the same transformation property (9) as  $\tilde{L}$ . Let us consider an action for these link variables

$$S_L = \sum_{\text{links}} W_L(L(x; \mu)) + S_p. \quad (21)$$

Here  $S_p$  is given by eqs. (18), (15), (16) with  $\tilde{L}$  replaced by  $L$ . It appears in  $S_L$  with a positive sign, in contrast to the negative sign in  $\tilde{S}_p = A_p - S_p$ . The “link potential”  $W_L$  depends only on the matrix  $L$  for one given link position  $(x; \mu)$ . We will use

$$W_L(L) = -\mu^2 \rho + \frac{\lambda_1}{2} \rho^2 + \frac{\lambda_2}{2} \tau_2, \\ \rho = \text{tr}(L^\dagger L), \quad \tau_2 = \frac{N}{2} \text{tr}(L^\dagger L - \frac{1}{N} \rho)^2, \quad (22)$$

where we assume  $\lambda_1 > 0$  and  $\lambda_2 > 0$  such that  $S_L$  is bounded from below. (Higher order terms could be added, if necessary.) A functional integral over link variables  $L$  with an action of the type (21) defines a model of “linear lattice gauge theory”. In contrast to the more standard “non-linear lattice gauge theory”, where the matrices  $L$  are replaced by unitary matrices  $U$  which obey the constraint  $U^\dagger U = 1$ , the matrices  $L$  are unconstrained. Standard lattice gauge theories with Wilson action are recovered if one replaces  $L$  by  $U$ . (Note that  $W_L$  becomes an irrelevant constant in this case.)

The relation between linear and non-linear lattice gauge theories is similar to the relation between linear and non-linear  $\sigma$ -models [12]. Consider a potential  $W_L(L)$  that takes its minimum for a unit matrix,  $L_0 = l_0 \mathbb{1}$ . This can be realized for positive  $\mu^2, \lambda_1$  and  $\lambda_2$ , with  $\rho_0 = N l_0^2 = \mu^2 / \lambda_1$ . We will next show that excitations around such a minimum describe a standard lattice gauge theory with unitary link

variables coupled to additional fields in the singlet and adjoint representations of the gauge group.

We can represent a complex  $N \times N$  matrix  $L$  as a product of a hermitean matrix  $S$  and a unitary matrix (polar decomposition)

$$L(x; \mu) = S(x; \mu) U(x; \mu), \quad S^\dagger = S, \quad U^\dagger U = 1. \quad (23)$$

The gauge transformation property

$$S'(x; \mu) = V(x) S(x; \mu) V^\dagger(x), \\ U'(x; \mu) = V(x) U(x; \mu) V^\dagger(x + e_\mu), \quad (24)$$

implies for  $U(x; \mu)$  the same transformation property as for  $L(x; \mu)$ , while  $S(x; \mu)$  involves only the gauge transformations at  $x$ . The fields

$$S(x; \mu) = l(x; \mu) + A(x; \mu), \\ l = \frac{1}{N} \text{tr} S, \quad \text{tr} A = 0, \quad A^\dagger = A, \quad (25)$$

decompose into a singlet  $l(x; \mu)$  and an adjoint representation  $A(x; \mu)$ . The singlet is invariant, while  $A$  transforms homogeneously with respect to local gauge transformations at the point  $x$ . For each site  $x$  we have  $d$  such fields, one for each value of the index  $\mu$  in  $S(x; \mu)$ . The precise properties of these fields with respect to the lattice symmetries are complicated. For example,  $\pi/2$ -rotations transform fields  $S(x; \mu)$  at different sites  $x$  into each other. Suitable averages of fields over the four  $\pi/2$ -rotations can be associated with scalar fields, while the differences from these averages belong to other representations of the discrete lattice rotation group. Such differences between fields  $S(x; \mu)$  add substantial complication without involving qualitatively new aspects. We may neglect them here and concentrate on  $S(x; \mu) = S(x; \nu) = S(x)$ , where  $S(x)$  is associated with a scalar field.

The matrices  $U(x; \mu)$  play the role of unitary link variables which are familiar in lattice gauge theories. With  $LL^\dagger = SS^\dagger$  the link potential is independent of  $U$ , i.e.  $W_L(L(x; \mu)) = W_L(S(x; \mu))$ . The unitary link variables appear only in the kinetic term  $\mathcal{L}_p$  through the invariant  $\tilde{P}$ . For the action (21) this implies  $S_L = S_g + S_W + S_A$ , with

$$S_g = \sum_{\text{plaquettes}} \{ l^2(x) l(x + e_\mu) l(x + e_\nu) \text{Re}(P_U(x; \mu, \nu)) \\ - \frac{N}{4} [l^4(x) + l^2(x) l^2(x + e_\mu) + l^2(x) l^2(x + e_\nu) \\ + l^2(x + e_\mu) l^2(x + e_\nu)] \}. \quad (26)$$

Here  $P_U(x; \mu, \nu)$  corresponds to  $\tilde{P}$  in eq. (10) with the replacement  $\tilde{L} \rightarrow U$ . For  $l(x) = l_0$  the “gauge part” of the action  $S_g$  is precisely the plaquette action of standard lattice gauge theories [1]

$$S_g = -\frac{2}{g^2} \sum_{\text{plaquettes}} \{ \text{Re} P_U(x; \mu, \nu) - N \}, \\ \frac{2}{g^2} = \frac{\beta}{3} = l_0^4. \quad (27)$$

In addition,  $S_g$  contains derivative terms for the scalar singlet  $l(x)$ .

The potential part

$$S_W = d \sum_x W_L[l(x) + A(x)] \quad (28)$$

involves the scalar fields  $l$  and  $A$ . Finally, the part  $S_A$  contains covariant kinetic terms for the adjoint scalar  $A$ . It arises from  $S_p$  and vanishes for  $A = 0$ . We conclude that for complex  $L$  and gauge group  $SU(N) \times U(1)$  the action of linear lattice gauge theory describes gauge fields as well as scalars in the adjoint and singlet representations. Similarly, for real  $L$  and gauge group  $SO(N)$  the matrices  $U$  are orthogonal,  $U^T U = 1$ , and  $A$  corresponds to a traceless symmetric tensor representation.

Let us now choose a potential  $W_L(S)$  for which a quadratic expansion around the minimum at  $S = l_0$ ,

$$W_L(S) = W_0 + \frac{1}{2} \bar{m}_l^2 l_0^2 (l - l_0)^2 + \frac{1}{2} \bar{m}_A^2 l_0^2 \text{tr}(A^2) + \dots, \\ \bar{m}_l^2 = 4N^2 \lambda_1, \quad \bar{m}_A^2 = 2N \lambda_2, \quad (29)$$

is characterized by large positive values  $\bar{m}_l^2 \gg 1, \bar{m}_A^2 \gg 1$ . Comparing with typical kinetic terms in the action

$$S_{\text{kin}}^{(l,A)} = \sum_x \frac{1}{2} Z_l l_0^2 a^2 \partial_\mu l(x) \partial_\mu l(x) \\ + \frac{1}{2} Z_A l_0^2 a^2 \text{tr}(\partial_\mu A(x) \partial_\mu A(x)) \quad (30)$$

the normalized mass terms read in the continuum limit  $m_l^2 = \bar{m}_l^2 / (Z_l a^2), m_A^2 = \bar{m}_A^2 / (Z_A a^2)$ . For very large  $\bar{m}_l^2$  and  $\bar{m}_A^2$  the fluctuations of the scalar fields are strongly suppressed and give only minor corrections to the functional integral. In the limit  $\bar{m}_l^2 \rightarrow \infty, \bar{m}_A^2 \rightarrow \infty$  we expect linear lattice gauge theory to give the same results as non-linear lattice gauge theory for the corresponding value of  $\beta = 3l_0^4$ . This extends to the more complicated structure of fields  $S(x; \mu)$ . We conclude that our model has a simple limit. For  $\lambda_1 \rightarrow \infty, \lambda_2 \rightarrow \infty, \mu^2 = N \lambda_1 l_0^2 \rightarrow \infty$ , with fixed  $l_0^2$ , the linear lattice gauge theory is equivalent to the standard (non-linear) lattice gauge theory with  $\beta = 3l_0^4$ . (Additional terms in the action involving the invariant  $D$  discussed in appendix A, e.g. eq. (A.2), can be used to change the ratio of  $SU(N)$ - and  $U(1)$ -gauge couplings, or to remove the links for the abelian  $U(1)$  field from the low energy spectrum.)

Starting from the limit  $\lambda_{1,2} \rightarrow \infty$  we may lower the values of the couplings  $\lambda_1$  and  $\lambda_2$  while keeping  $l_0^2$  fixed. For smaller values of  $\bar{m}_l^2$  and  $\bar{m}_A^2$  we still expect the model to be in the same universality class as standard lattice gauge theories. The long distance behavior will be characterized by the value of the renormalized gauge coupling. Its precise relation to the microscopic gauge couplings  $g$  can typically be influenced by the presence of scalar fluctuations with masses of the order of the inverse lattice distance.

The microscopic lattice gauge coupling is given by the inverse of the fourth power of the expectation value  $l_0$

$$g^2 = \frac{2}{l_0^4}. \quad (31)$$

It is small for large  $l_0$  such that lattice perturbation theory can be applied for not too large distance scales. On the other hand, for small  $l_0$  one has a large  $g^2$  and a strong coupling expansion becomes valid. For a simple non-abelian gauge group (as  $SO(2N)$  or  $SU(N)$ ) all correlation functions are expected to decay exponentially in this case, and no non-trivial long distance behavior is expected. (If the gauge group has an abelian factor (as  $SU(N) \times U(1)$ ) non-trivial long distance behavior may be associated to a Coulomb type interaction in the abelian sector.)

The situation is analogous to the relation between the linear and non-linear non-abelian  $\sigma$ -models in two dimensions [13–15]. It is worthwhile to recall the properties of these models since important conclusions for four dimensional gauge theories can be drawn. The coupling of the non-linear  $\sigma$ -model is given by the inverse of the expectation value  $\langle \varphi \rangle = \kappa$  for which the potential  $V(\varphi)$  in the linear model takes its minimum,  $g^2 = (2\kappa)^{-1}$ . Including the effect of fluctuations the microscopic couplings are replaced by running renormalized couplings. Then the running of  $g^2$  in the non-linear model can be associated to the running of  $\kappa$  in the linear model. One can study the fluctuation induced change of the effective potential  $V(\varphi)$  in the linear model by use of functional renormalization [13–15]. For this purpose one introduces an effective infrared cutoff  $k$  in order to define the effective average action  $\Gamma_k$  which includes the quantum fluctuations with momenta larger than  $k$ . The scale dependence of  $\Gamma_k$ , and correspondingly of the effective average potential  $V_k(\varphi)$ , is governed by an exact functional differential equation with one loop structure [16]. For a non-abelian  $SO(N)$  symmetry and large  $\kappa$  one obtains in leading order of a derivative expansion for the  $k$ -dependence of the minimum of  $V_k(\varphi)$  the flow equation

$$k \partial_k \kappa = \frac{N-2}{4\pi}. \quad (32)$$

With  $g^2 = (2\kappa)^{-1}$  this reproduces precisely the one loop result for the running of  $g$  in the non-linear  $\sigma$ -model.

Starting at some ultraviolet scale  $\Lambda$  with  $\kappa_\Lambda$  eq. (32) implies that  $\kappa(k)$  vanishes for a scale  $k_s = \exp(-\kappa_\Lambda) \Lambda$ . This feature persists beyond the leading order in the derivative expansion [15]. For  $k < k_s$  the minimum of  $V_k(\varphi)$  is at  $\varphi = 0$ . Once all quantum fluctuations are included for  $k \rightarrow 0$  no spontaneous symmetry breaking is present, in accordance with the Mermin-Wagner theorem [17]. The strong coupling regime of the non-linear  $\sigma$ -model is simply described by the symmetric regime of the linear  $\sigma$ -model.

One may wonder if a similar simple description for the confinement regime of four-dimensional Yang-Mills theories is possible within linear gauge theories. One may choose a normalization for the singlet field such that the continuum limit for the average action becomes for all  $k$

$$\Gamma_k = \int_x \frac{1}{4} l^4 F_{\mu\nu}^z F_{\mu\nu}^z + W_k(l) + \dots, \quad (33)$$

where  $F_{\mu\nu}^z$  is the non-abelian field strength for the gauge fields and the dots denote terms involving derivatives  $\partial_\mu l$ , higher derivatives of the gauge field or additional fields. The one loop running for the gauge coupling,

$\partial g/\partial t = -\bar{\beta}g^3$ , translates to the flow of the minimum of  $W_k$  at  $l_0(k)$

$$k\partial_k l_0 = \frac{\bar{\beta}}{l_0^3}. \quad (34)$$

This equation is valid for large  $l_0$  and large  $\lambda_1, \lambda_2$ .

The strong coupling regime corresponds to  $l_0(k)$  approaching zero. The crucial question for an understanding of confinement is then what other terms beyond eq. (33) may become important for vanishing or very small  $l_0$ . A natural candidate is a description in terms of continuous bilocal linear link fields  $L(x, y)$ . The confinement regime corresponds to a “link potential” for which the minimum occurs for  $\rho(x, y) = \text{tr}\{L^\dagger(x, y)L(x, y)\} = 0$  if  $(y - x)$  exceeds a characteristic length scale. We will turn to this issue in a separate publication.

#### IV. GAUGE BOSONS AS COLLECTIVE EXCITATIONS

In this section we establish the connection between linear lattice gauge theory and scalar lattice gauge theory discussed in sect. II. The link variables  $L(x; \mu)$  in sect. III will be related to collective fields for scalar bilinears  $\tilde{L}(x; \mu)$ . By means of a Hubbard-Stratonovich type transformation we will discuss a functional integral for scalars and link variables that is equivalent to the scalar lattice gauge theory presented in sect. II. In particular, we will argue in sect. V that scalar lattice gauge theory is equivalent to pure standard lattice gauge theory for an appropriate limit  $\bar{\lambda} \rightarrow \infty$  for the action (5).

The appropriate formalism for collective fields introduces sources  $K_{ij}(x; \mu)$  for the link bilinears  $\tilde{L}_{ij}(x; \mu)$  in addition to the sources  $j_i^a(x)$  for the “fundamental scalars”  $\chi_i^a(x)$

$$Z[j, K] = \exp W[j, K] = \int \mathcal{D}\chi \exp \{ -S[\chi] + \sum_x [(j_i^a)^* \chi_i^a + \sum_\mu K_{ij}^*(x; \mu) \tilde{L}_{ij}(x; \mu) + c.c.] \}. \quad (35)$$

The definition (8) for the link variables implies an identity for the functional dependence of  $W$  on  $K$  and  $j$ ,

$$\begin{aligned} \langle \tilde{L}_{ij}(x; \mu) \rangle &= \frac{\partial W}{\partial K_{ij}^*(x; \mu)} \\ &= \frac{\partial^2 W}{\partial (j_i^a(x))^* \partial j_j^a(x + e_\mu)} + \frac{\partial W}{\partial (j_i^a(x))^*} \frac{\partial W}{\partial j_j^a(x + e_\mu)}. \end{aligned} \quad (36)$$

Here  $\langle \tilde{L}_{ij}(x; \mu) \rangle$  denotes the expectation value of the link bilinear in the presence of arbitrary sources  $j, K$ . We define

$$\varphi_i^a(x) = \langle \chi_i^a(x) \rangle = \frac{\partial W}{\partial (j_i^a(x))^*} \quad (37)$$

and (“background”) link variables

$$\bar{L}_{ij}(x; \mu) = \langle \tilde{L}_{ij}(x; \mu) \rangle. \quad (38)$$

The quantum effective action  $\Gamma[\varphi, \bar{L}]$  obtains from  $W[j, K]$  by the usual Legendre transform

$$\begin{aligned} \Gamma[\varphi, \bar{L}] &= -W[j, K] + \sum_x [\text{tr}\{j^\dagger(x)\chi(x)\} \\ &+ \sum_\mu \text{tr}\{K^\dagger(x; \mu)\bar{L}(x; \mu)\} + c.c.], \end{aligned} \quad (39)$$

where we use the matrix notation. The sources  $j$  and  $K$  are expressed in terms of  $\varphi$  and  $\bar{L}$  by inverting the relations (36), (37), resulting in

$$\frac{\partial \Gamma}{\partial \varphi(x)} = j^\dagger(x), \quad \frac{\partial \Gamma}{\partial \bar{L}(x)} = K^\dagger(x). \quad (40)$$

The identity (36) translates into a corresponding identity for  $\Gamma$  that relates its dependence on  $\bar{L}$  to its dependence on  $\varphi$  [18]. A computation of the effective action  $\Gamma[\varphi, \bar{L}]$  would establish the connection between scalar lattice gauge theory and linear lattice gauge theory on a “macroscopic level” where fluctuation effects are all included. In this section we will aim for a more microscopic relation where bilinears  $\tilde{L}(x; \mu)$  are related to fluctuating fields  $L(x; \mu)$ . The price to pay is a more complicated relation between the expectation values  $\langle L(x; \mu) \rangle$  and  $\langle \tilde{L}(x; \mu) \rangle$  and corresponding higher order correlation functions.

Linear lattice gauge theory is formulated as a functional integral over link variables. In scalar lattice gauge theory this link-integration can be implemented via a Hubbard-Stratonovich type transformation [10, 11]. For this purpose one uses the identity

$$\begin{aligned} \int_{-\infty}^{\infty} dL_{ij}(x; \mu) \exp \{ -f[L_{ij}(x; \mu) \\ - \chi_i(x)\chi_j^*(x + e_\mu)] \} = c, \end{aligned} \quad (41)$$

which holds for arbitrary functions  $f$  as long as  $|L_{ij}| \rightarrow \infty$  implies  $f(L_{ij}) \rightarrow \infty$ . The constant does not depend on fields or sources, as can be seen easily by a shift of the integration variable. A similar argument allows us to insert into the functional integral (2) the expression

$$\int \mathcal{D}L \exp \{ -\bar{S}_L[L_{ij}(x; \mu) - \chi_i(x)\chi_j^*(x + e_\mu)] \} = Z_L, \quad (42)$$

where the functional integration  $\int \mathcal{D}L$  corresponds to a product over all links of integrations over individual link variables. The field independent constant  $Z_L$  is an irrelevant multiplicative renormalization of  $Z$ , or additive renormalization of  $W$  and  $\Gamma$ , and it may be dropped. The “link action”  $\bar{S}_L$  can be chosen arbitrarily as long as the integral  $Z_L$  is well defined and  $\bar{S}_L \rightarrow \infty$  for  $|L_{ij}(x; \mu)| \rightarrow \infty$ .

A possible choice takes for  $\bar{S}_L$  the action  $S_L$  in eq. (21). Since  $S_L$  is not quadratic in the fields and has a minimum for fields different from zero the transformation (42) will introduce several new features as compared to the Hubbard-Stratonovich transformation. It will help us,

however, to understand the close connection between scalar lattice gauge theory and linear lattice gauge theory.

Insertion of eq. (42) into the scalar functional integral (35) yields a lattice model for link variables and scalar variables. The microscopic action for this “link-scalar model” consists of four pieces,

$$S[\chi, L] = S_\chi + S_L + \Delta S_\chi + S_{\text{int}} : \quad (43)$$

(i) The scalar action  $S_\chi[\chi]$  is the original action of scalar lattice gauge theory in eq. (35). We take it in the form

$$S_\chi[\chi] = S_V[\chi] + S_{\text{kin}}[\chi] + \sum_{\text{links}} W_\chi[\tilde{L}] + \sum_{\text{plaquettes}} (\mathcal{A}_p[\chi] - \mathcal{L}_p[\tilde{L}]). \quad (44)$$

The last term corresponds to  $\sum \mathcal{S}_p$  in eqs. (7), (19) with  $\mathcal{L}_p$  and  $\mathcal{A}_p$  given by eqs. (17) and (20). The potential term  $S_V$  (cf. eq. (A.3))

$$S_V = \sum_x V_\chi(\chi(x)), \quad (45)$$

and kinetic term  $S_K$  (similar to eq. (A.5)), as well  $W_\chi[\tilde{L}]$ , will be determined later. (ii) The link action  $S_L[L]$  is the part of  $S_L$  in eq. (42) that only involves  $L$ , while (iii) the shift in the scalar action  $\Delta S_\chi[\chi]$  is the corresponding part from eq. (42) that only involves  $\chi$ . (iv) Finally, there is a coupling between the links and the scalars,

$$S_{\text{int}}[\chi, L] = S_L[L_{ij}(x; \mu) - \tilde{L}_{ij}(x; \mu)] - S_L[L_{ij}(x; \mu)] - S_L[\tilde{L}_{ij}(x; \mu)]. \quad (46)$$

The plaquette term  $S_p[\tilde{L}]$  in  $\Delta S_\chi$  cancels the plaquette term  $-S_p[\tilde{L}]$  in  $S_\chi$ . This eliminates the most cumbersome derivative interaction in scalar lattice gauge theory in favor of a plaquette action for links. Furthermore, by the use of identities of the type (A.7) or (A.12) we can always write  $S_\chi$  in the form (44) with  $W_\chi[\tilde{L}] = -W_L[-\tilde{L}]$ . Thus  $W_\chi[\tilde{L}]$  is canceled by the contribution  $W_L[-\tilde{L}]$  from  $\Delta S_\chi$ . The microscopic action for scalars and links becomes then

$$S[\chi, L] = S_V[\chi] + S_{\text{der}}[\chi] + S_L[L] + S_{\text{int}}[\chi, L], \quad (47)$$

where the derivative terms for  $\chi$  include interactions

$$S_{\text{der}}[\chi] = S_{\text{kin}}[\chi] + \sum_{\text{plaquettes}} \mathcal{A}_p. \quad (48)$$

This action describes linear lattice gauge theory coupled to scalars.

The remaining derivative part  $S_{\text{der}}[\chi]$  is much simpler than the derivative part in the original action  $S_\chi$  of scalar lattice gravity. As we have discussed in sect. II the invariant  $\mathcal{A}_p$  suppresses the contributions of highly inhomogeneous scalar bilinears  $M(x)$  to the functional integral. The price to pay is the presence of explicit link variables for the gauge field and a relatively complex interaction term. A simpler picture is only realized if the role of the interaction

term is subleading, and we will discuss conditions for this in the next section.

Indeed, the coupling between links and scalars is rather complicated. The piece  $S_{\text{int}}$  involves terms with up to three powers of  $L$  and up to six powers of  $\chi$ . The terms quadratic in  $\chi$  obtain from a Taylor expansion of  $S_L[L]$

$$\begin{aligned} S_{\text{int}}^{(2)} &= - \sum_{\text{links}} \left\{ \frac{\partial S_L[L]}{\partial L_{ij}(x; \mu)} \tilde{L}_{ij}(x; \mu) \right. \\ &\quad \left. + \frac{\partial S_L[L]}{\partial L_{ij}^\dagger(x; \mu)} \tilde{L}_{ij}^\dagger(x; \mu) \right\} \\ &= - \sum_{x, \mu} \chi_j^{a*}(x + e_\mu) \frac{\partial S_L}{\partial L_{ij}(x; \mu)} \chi_i^a(x) + c.c.. \end{aligned} \quad (49)$$

In particular, this quadratic term vanishes whenever  $L$  corresponds to an extremum of  $S_L[L]$ . Terms with four powers of  $\chi$  are proportional to the second  $L$ -derivative of  $S_L$ , while six powers of  $\chi$  multiply the third derivative.

As an example, the contribution  $\sim \mu^2$  reads

$$S_{\text{int}}^{(\mu^2)} = \mu^2 \sum_{\text{links}} \text{tr} \{ \chi^\dagger(x + e_\mu) L^\dagger(x; \mu) \chi(x) \} + c.c.. \quad (50)$$

If we define a covariant derivative  $D_\mu$  by

$$a D_\mu \chi(x) = L(x; \mu) \chi(x + e_\mu) - \chi(x) \quad (51)$$

we can write invariants of the type

$$\begin{aligned} \text{tr} \{ (D_\mu \chi(x))^\dagger D_\mu \chi(x) \} &= \\ \text{tr} \{ \chi^\dagger(x + e_\mu) L^\dagger(x; \mu) L(x; \mu) \chi(x + e_\mu) & \\ + \chi^\dagger(x) \chi(x) - (\chi^\dagger(x + e_\mu) L^\dagger(x; \mu) \chi(x) + c.c.) \}. \end{aligned} \quad (52)$$

We can associate the interaction (50) with the interaction between two scalars and a link that appears in the squared covariant derivative.

We next discuss more specifically the action (44) for scalar lattice gauge theory that is equivalent to the link-scalar action (47). This requires to express  $W_\chi[\tilde{L}]$  in terms of  $\chi$  and to specify  $S_V$  and  $S_{\text{kin}}$ . For the choice (22) one has

$$\begin{aligned} W_\chi(\tilde{L}) &= -W_L(-\tilde{L}) = -W_L(\tilde{L}) \\ &= \mu^2 \tilde{\rho} - \frac{\lambda_1}{2} \tilde{\rho}^2 - \frac{\lambda_2}{2} \tilde{\tau}_2. \end{aligned} \quad (53)$$

Here  $\tilde{\rho}$  and  $\tilde{\tau}_2$  obey eq. (22) with the replacement  $L \rightarrow \tilde{L}$ . In terms of the scalar matrix  $M$  they read

$$\begin{aligned} \tilde{\rho}(x; \mu) &= \text{tr} \{ M(x) M(x + e_\mu) \}, \\ \tilde{\tau}_2(x; \mu) &= \frac{N}{2} \text{tr} \left\{ [M(x) M(x + e_\mu) \right. \\ &\quad \left. - \frac{1}{N} \text{tr} \{ M(x) M(x + e_\mu) \}]^2 \right\}. \end{aligned}$$

We write

$$\sum_{\text{links}} W_\chi[\tilde{L}] = S_{W, \text{der}} + S_{W, l} - \sum_x V_W(x) \quad (54)$$

with

$$S_{W,\text{der}} = \frac{a^2}{2} \sum_x \sum_\mu \left[ \left( \frac{\lambda_1}{4} - \frac{\lambda_2}{8} \right) \right. \\ \times \left\{ \left[ \text{tr} \{ [M(x) + M(x + e_\mu)] \partial_\mu M(x) \} \right]^2 \right. \\ \left. + \text{tr} \{ [M(x) + M(x + e_\mu)]^2 \} \text{tr} \{ \partial_\mu M(x) \partial_\mu M(x) \} \right\} \\ \left. + \frac{\lambda_2 N}{4} \text{tr} \{ [M(x) + M(x + e_\mu)]^2 \partial_\mu M(x) \partial_\mu M(x) \} \right], \quad (55)$$

and

$$S_{W,l} = \sum_{\text{links}} \frac{\mu^2}{2} \text{tr} \{ [M(x + e_\mu) + M(x)]^2 \}. \quad (56)$$

The corresponding potential term reads

$$V_W(\chi) = d\mu^2 \text{tr} M^2(x) + \frac{d}{2} (\lambda_1 - \frac{1}{2} \lambda_2) (\text{tr} M^2(x))^2 \\ + \frac{dN\lambda_2}{4} \text{tr} M^4(x). \quad (57)$$

Defining for  $S_V$  in eq. (44), (45)

$$V_\chi(\chi) = \bar{V}(\chi) + V_W(\chi), \quad (58)$$

we arrive at the action (44) for scalar lattice gauge theory

$$S_\chi[\chi] = \sum_x \bar{V}(\chi(x)) + S_{W,l} + \bar{S}_{\text{der}}, \quad (59)$$

with derivative term composed from three pieces

$$\bar{S}_{\text{der}} = S_{\text{kin}}[\chi] + S_{W,\text{der}} + \bar{S}_p. \quad (60)$$

For a given choice of  $\bar{V}(\chi)$  and  $S_{\text{kin}}$  this fixes the action  $S_\chi[\chi]$  of lattice scalar gravity completely. Eqs. (59), (60) are our final result for the choice of  $S_\chi$ . We observe that  $S_{W,l}$  can be written as a sum of potential and derivative terms for  $M$ .

At this point we may compare with the action (5) in sect. II. The term  $\bar{S}_p$  is common. In  $S_{W,\text{der}}$  the last term equals the first term in eq. (6) if we identify

$$\bar{\lambda} = \frac{\lambda_2 N}{2} + \frac{d-1}{2}. \quad (61)$$

Furthermore  $S_{W,\text{der}}$  contains an additional derivative term  $\sim (\lambda_1 - \lambda_2/2)$  that we have omitted in sect. II (e.g. setting  $\lambda_1 = \lambda_2/2$ ). The contribution  $S_{W,l}$  equals the second term in eq. (6) provided we take

$$\bar{m}^2 = 2d\mu^2 a^{-\frac{d+2}{2}}. \quad (62)$$

In summary, the action (44) of scalar lattice gauge theory equals the action (5), plus additional pieces

$$S' = \sum_x \bar{V}(\chi(x)) + S_{\text{kin}} + \Delta S_{\text{der}}, \quad (63)$$

with  $\Delta S_{\text{der}}$  the term in  $S_{W,\text{der}}$  proportional  $(\lambda_1 - \lambda_2/2)$ . For a simplified discussion we may take  $S' = 0$ , such that

eqs. (5) and (59) coincide. Scalar lattice gauge theory with action  $S_\chi[\chi]$  given by eq. (5) is then completely equivalent to the link-scalar model with functional integral over links and scalars and action  $S[\chi, L]$  given by eq. (47).

At this stage the collective source  $K$  multiplies only the bilinear  $\tilde{L}$  according to eq. (35). A coupling of  $K$  to the link variables  $L$  can be implemented by a modification of the choice  $\tilde{S}_L = S_L[L - \tilde{L}]$  to  $\tilde{S}_L = S_L[\tilde{L} - \hat{K}]$ , with  $\hat{K}$  a linear combination of the different representations in  $K$ ,

$$K = k_R + ik_I + K_A + iK_B, \\ \hat{K} = c_R k_R + ic_I k_I + c_A K_A + ic_B K_B. \quad (64)$$

Here  $c_R, c_I, c_A, c_B$  are real constants,  $k_R$  and  $k_I$  are proportional to the unit matrix, and  $\text{tr} K_A = \text{tr} K_B = 0$ ,  $K_A^\dagger = K_A$ ,  $K_B^\dagger = K_B$ . The factor  $Z_L$  in eq. (42) is the same, such that the model is not changed for an arbitrary choice of coefficients  $c_j$ . The functionals  $W[j, K]$  and  $\Gamma[\varphi, \tilde{L}]$  are the same for scalar lattice gauge theory and the link-scalar model obtained from the  $K$ -dependent modification of the transformation (42). However, the association of  $K$ -derivatives of  $W$  with expectation values and correlation functions depends on the choice of  $c_j$ .

This can be seen by expanding  $S_L[L - \tilde{L} - \hat{K}]$  around its minimum (omitting an irrelevant constant) at  $L - \tilde{L} - \hat{K} = l_0$ . With the decomposition

$$L = l + is + A + iB, \\ l = \frac{1}{2N} \text{tr} \{ L + L^\dagger \}, \quad s = -\frac{i}{2N} \text{tr} \{ L - L^\dagger \}, \\ A = \frac{1}{2} (L + L^\dagger) - l, \quad B = -\frac{i}{2} (L - L^\dagger) - s, \\ \text{tr} A = \text{tr} B = 0, \quad A^\dagger = A, \quad B^\dagger = B, \quad (65)$$

and similar for  $\tilde{L}$ , one finds

$$S_L = \sum_{\text{links}} 2N^2 \lambda_1 l_0^2 (l - \tilde{l} - l_0 - c_R k_R)^2 \\ + N \lambda_2 l_0^2 \text{tr} \{ (A - \tilde{A} - c_A K_A)^2 \} + S_{L,\text{der}}^{(2)} + \dots \quad (66)$$

where  $S_{L,\text{der}}^{(2)}$  contains derivative terms from an expansion of  $S_p$  in quadratic order in  $(L - \tilde{L} - \hat{K} - l_0)$ . Neglecting the derivative terms and higher orders we find a contribution of  $S_L$  linear in  $k_R$  and  $K_A$

$$S_{L,K}^{(1)} = -4N^2 \lambda_1 l_0^2 c_R k_R (l - \tilde{l} - l_0) \\ - 2N \lambda_2 l_0^2 c_A \text{tr} \{ K_A (A - \tilde{A}) \}. \quad (67)$$

This yields for every link  $(x; \mu)$

$$\frac{\partial W}{\partial k_R} = 2N \langle \tilde{l} + 2N \lambda_1 l_0^2 c_R (l - \tilde{l} - l_0) \rangle, \\ \frac{\partial W}{\partial K_A} = 2 \langle \tilde{A} + N \lambda_2 l_0^2 c_A (A - \tilde{A}) \rangle, \quad (68)$$

to be compared with the equivalent eq. (36) which corresponds to  $c_R = c_A = 0$ . For  $c_j \neq 0$  the expectation value  $\langle \tilde{L} \rangle_\chi$  in the scalar lattice gauge theory does not equal the



value  $\langle \tilde{L} \rangle_{\chi L}$  in the equivalent link-scalar model. This is due to the additional expectation value  $\langle L \rangle_{\chi L}$  of the link variable. In particular, for the choice  $c_R = (2N\lambda_1 l_0^2)^{-1}$  the collective source  $k_R$  decouples from the link bilinear  $\tilde{l}$  and one obtains the leading order relation

$$\langle \tilde{l} \rangle_{\chi} = \langle l \rangle_{\chi L} - l_0. \quad (69)$$

Similarly, for  $c_A = (N\lambda_2 l_0^2)^{-1}$  the source  $K_A$  decouples from  $\tilde{A}$  and one has

$$\langle \tilde{A} \rangle_{\chi} = \langle A \rangle_{\chi L}. \quad (70)$$

The relations (69), (70) get modified by higher orders in the expansion of  $S_L$  in powers of  $(L - \tilde{L} - l_0 - \hat{K})$ . The terms linear in  $K$  add on the r.h.s. expectation values  $\sim \langle (L - \tilde{L} - l_0)^n \rangle$ ,  $n = 2, 3$ . Higher powers in  $K$  in the expansion of  $S_L[L - \tilde{L} - \hat{K}]$  will influence the precise relation between correlation functions in scalar lattice gauge theory and the equivalent link-scalar model. The situation is similar for the sources  $k_I$  and  $K_B$  for which the additional contributions arise from  $S_p$ . In Fourier space the coefficients  $c_j$  can be taken as functions of the squared momentum.

The upshot of these considerations shows that the correlation functions for the link variables  $L$  in the link-scalar model have similar properties as corresponding correlation functions for  $\tilde{L}$  in scalar lattice gauge theories. The detailed relation is rather involved, however, and reflects operator mixing for observables with the same transformation properties. Despite these complications the expectation values of Wilson loops for  $L$  can be used for confinement criteria similar to standard lattice gauge theories.

We emphasize that the choice  $\tilde{S}_L = S_L$  is only one particular possibility to introduce an integration over link variables. Many other choices are possible and may be convenient in order to obtain a simpler form for the interaction  $S_{\text{int}}$ . As an example, consider

$$\tilde{S}_L = S_{LK} = \sum_{\text{links}} \text{tr}(L^\dagger - \tilde{L}^\dagger - K^\dagger)(L - \tilde{L} - K). \quad (71)$$

This results in

$$\begin{aligned} S_L &= \sum_{\text{links}} \text{tr}\{L^\dagger L\}, \quad S_{\text{int}} = - \sum_{\text{links}} \text{tr}\{L^\dagger \tilde{L} + \tilde{L}^\dagger L\}, \\ \Delta S_\chi &= \sum_{\text{links}} \text{tr}\{\tilde{L}^\dagger \tilde{L}\}. \end{aligned} \quad (72)$$

Furthermore, the term  $\text{tr}\{(\tilde{L}^\dagger - L^\dagger)K + K^\dagger(\tilde{L} - L)\}$  replaces in eq. (35) the source term for the bilinear  $K^\dagger \tilde{L}$  by a standard linear source term for the links  $K^\dagger L$ , resulting in  $\langle L \rangle_{\chi L} = \langle \tilde{L} \rangle_{\chi}$ . (The additional term in  $W \sim \text{tr} K^\dagger K$  has to be taken into account if correlation functions are computed from derivatives of  $W$ .)

With the choice (71) the interaction becomes a simple cubic interaction involving one link and two scalar fields  $\sim \text{tr} L^\dagger \chi \chi^\dagger$ . The microscopic link action contains only a quadratic link potential. In particular, there is no plaquette term  $\mathcal{L}_p$  for the links, while the cumbersome derivative interaction  $-\mathcal{L}_p(M)$  is still present in the scalar sector.

Nevertheless, the two choices  $\tilde{S}_L = S_L$  or  $\tilde{S}_L = S_{LK}$  are completely equivalent. For the choice  $\tilde{S}_L = S_{LK}$  an effective plaquette term for the link variables  $L$  will be induced by quantum fluctuations.

## V. PHASE DIAGRAM OF SCALAR LATTICE GAUGE THEORY

Scalar lattice gauge theory with collective link variables describes gauge bosons, scalars in the singlet and adjoint representation (from  $L$ ) as well as flavored scalars in the fundamental representation  $\chi$ . Depending on the choice of the parameters in the action  $S_\chi$  one expects a rich phase diagram, with the gauge sector in the confined or Higgs-phase, or without any long range interactions at all. The Higgs phase describes loosely speaking spontaneous symmetry breaking by expectation values of  $\chi$  or  $A$ . (Spontaneous symmetry breaking of a local gauge symmetry occurs only in a gauge fixed version, being absent in a gauge invariant formulation. The relation between the different pictures is well known.)

It is not our aim to explore this phase diagram systematically in this paper. We mainly want to argue that for a suitable choice of  $S_\chi$  one can realize the standard confinement phase of QCD. For this purpose we present a recipe how to construct an action  $S_\chi$  for scalar lattice gauge theory that is equivalent to the standard lattice gauge theory formulated in terms of unitary link variables.

We employ the formulation as a link-scalar model by taking in eq. (42) the choice  $\tilde{S}_L = S_L$ . The action is given by eq. (47),

$$\begin{aligned} S[\chi, L] &= \sum_x \left( \bar{V}(\chi(x)) + V_W(\chi(x)) \right) + S_{\text{der}}[\chi] \\ &\quad + S_L[L] + S_{\text{int}}[\chi, L]. \end{aligned} \quad (73)$$

The qualitative properties of our model will depend on the role of the interaction term  $S_{\text{int}}[\chi, L] = S_{\text{int}}[\tilde{L}, L]$  which is defined in eq. (46). Its possible importance can be seen from the identities

$$S_{\text{int}}[\tilde{L} = 0, L] = 0, \quad S_{\text{int}}[\tilde{L} = L, L] = -2S_L[L]. \quad (74)$$

The influence of the interaction term on the dynamics of the links remains small if the functional integral is dominated by configurations with small  $\chi$ . For simplicity we choose  $\lambda_1 = \lambda_2/2$ ,  $\bar{V} = 0$ ,  $S_{\text{kin}} = 0$  such that  $S' = 0$  in eq. (63). The action for the equivalent scalar lattice gauge theory is then given by eq. (5). It has two free parameters,  $\bar{\lambda}$  and  $\bar{m}^2$ , or, equivalently,  $\lambda_2$  and  $\mu^2$ . (The two parameter sets are related by eqs. (61), (62).)

Consider now the limit  $\lambda_2 \rightarrow \infty, \mu^2 \rightarrow \infty$  with fixed ratio

$$l_0^2 = \frac{\mu^2}{N\lambda_1} = \frac{2\mu^2}{N\lambda_2}. \quad (75)$$

The potential term  $V_W$  in eqs. (73), (57) reads

$$V_W = \frac{dN\lambda_2}{4} (2l_0^2 \text{tr} M^2 + \text{tr} M^4), \quad (76)$$

with minimum at  $M = 0$ . Also  $S_{\text{der}}[\chi]$ , given by eqs. (48), (20), has its minimum for  $M = 0$ . However, the potential term (75) will dominate for  $\lambda_2 \rightarrow \infty$ . The diverging quadratic term  $\sim \text{tr} M^2$  in eq. (76) will suppress strongly the fluctuations of  $M$ . On the other hand, the link action  $S_L$  takes its minimum for a homogeneous unit matrix  $L(x; \mu) = l_0$ . Again, the diverging mass terms (29) strongly suppress the fluctuations of  $l - l_0$  and  $A$ . Thus the dominant fluctuations around the minimum of  $S_L$  are the unitary link variables with

$$\begin{aligned} L &= l_0 U, \\ U &= \exp \left\{ \frac{i}{l_0} (s + B) \right\}. \end{aligned} \quad (77)$$

If the interaction term  $S_{\text{int}}$  is subleading, one expects to recover standard lattice gauge theory with unitary link variables and  $g^2$  or  $\beta$  given by eq. (27).

For an estimate of the importance of the interaction term  $S_{\text{int}}$  we first note that it involves only terms with even powers of  $\chi$ . This implies that the total action  $S[\chi, L]$  has an extremum for  $\chi = 0, L = l_0$ . We next study the quadratic terms for excitations around this extremum. The fields  $A, l - l_0$  and  $\chi$  do not mix since they belong to different representations of the gauge group. The quadratic terms for  $A$  and  $l - l_0$  are not affected by the interaction term. They are large according to eq. (29). In contrast, the quadratic term for  $\chi$  can receive contribution from  $S_{\text{int}}$ . By virtue of eq. (49) they vanish, however, for the extremum of  $S_L$  at  $L(x; \mu) = l_0$ . For a positive quadratic term from  $\bar{V}(\chi)$  we find that the homogeneous configuration  $\chi = 0, L = l_0$  is indeed a local minimum of the link-scalar action. This extends to  $\bar{V}(\chi) = 0$ . Now the dominant terms for small  $\chi$  are quartic in  $\chi$ . They correspond to the terms  $\sim \tilde{L}^2$  in the expansion of  $S_L[L - \tilde{L}] - S_L[\tilde{L}]$ . The leading terms arise from

$$\hat{W}_L = \frac{N\lambda_2}{4} \text{tr} \{ [(L - \tilde{L})^\dagger (L - \tilde{L})]^2 \}. \quad (78)$$

With  $L$  approximated by  $l_0$  this yields for the term  $\sim \chi^4$

$$\hat{W}_L^{(4)} = \frac{N\lambda_2}{4} l_0^2 \text{tr} \{ (\tilde{L}^\dagger + \tilde{L})^2 + 2\tilde{L}^\dagger \tilde{L} \}. \quad (79)$$

This term strongly suppresses fluctuations of  $\tilde{L}$  such that the role of the interaction term is indeed negligible. We conclude that the limit  $\lambda_2 \rightarrow \infty, l_0^2$  fixed, corresponds to standard lattice gauge theories with microscopic gauge coupling given by eq. (31).

We conclude that the limit

$$\bar{\lambda} \rightarrow \infty, \quad \bar{m}^2 = 2d\bar{\lambda}l_0^2 a^{-\frac{d+2}{2}} \quad (80)$$

of scalar lattice gauge theory with action (5) describes standard lattice gauge theory. Only the precise relation between correlation functions of unitary link variables in standard lattice gauge theories and correlation functions for link bilinears in scalar lattice gauge theory is complicated, as we have discussed in the preceding section. One expects that the same universality class extends to finite

large values of  $\bar{\lambda}$ . Other universality classes with additional light degrees of freedom may be realized for small  $\bar{\lambda}$  or in the presence of additional terms in the action of scalar lattice gauge theory.

While for large  $\bar{\lambda}$  a guess of the universality class seems rather straightforward in the formulation with scalar and link variables the situation is less obvious in the equivalent formulation which only uses scalars and action  $S_\chi$  as given by eq. (59). This formulation exhibits the derivative interactions given by  $S_{W\text{der}} + \bar{S}_p$ . It is not very clear a priori which is the dynamics induced by these derivative interactions which involve eight powers of  $\chi$  at up to four different lattice sites. The reformulation with link variables is therefore very helpful.

## VI. GAUGE INVARIANT FORMULATION

The action of scalar lattice gauge theory involves only the gauge invariant matrix  $M$ . One may therefore wish to reformulate the model in terms of a functional integral only involving the gauge invariant variables  $M$ . This gets rid of the gauge degrees of freedom altogether.

One can indeed achieve such a formulation by inserting in the functional integral an integration over a product of  $\delta$ -functions

$$\begin{aligned} \int \mathcal{D}M \prod_x \delta(M(x) - \chi^\dagger(x)\chi(x)) &= 1, \\ \int \mathcal{D}M &= \prod_x \int dM(x). \end{aligned} \quad (81)$$

Here  $\int dM$  denotes an integral over the  $M^2$  independent components of the hermitean  $M \times M$  matrix  $M(x)$ . Since the action only depends on  $M(x)$  this yields

$$\begin{aligned} Z &= \int \mathcal{D}\chi \exp \{ -S_\chi[M(\chi)] \} \\ &= \int \mathcal{D}M \exp \{ -S_\chi[M] - \sum_x V_M(M(x)) \}, \end{aligned} \quad (82)$$

with

$$V_M(M) = -\ln B(M), \quad B(M) = \int d\chi d\chi^\dagger \delta(M - \chi^\dagger \chi). \quad (83)$$

A few properties of  $B(M)$  can be found from simple arguments. The matrix  $\chi^\dagger \chi$  obeys constraints - for example, the diagonal elements cannot be negative. If  $M$  does not obey the same constraints a solution  $M = \chi^\dagger \chi$  does not exist and  $B(M) = 0$ . Thus the potential  $V(M)$  diverges if  $M$  does not obey the constraints, restricting effectively the allowed values of  $M$  to such matrices that can be represented as  $\chi^\dagger \chi$ . We next investigate matrices  $M$  for which  $B(M) \neq 0$ , and begin with diagonal matrices. From a rescaling  $\chi \rightarrow \alpha\chi, M \rightarrow \alpha^2 M$  one infers  $B(\alpha^2 M) = \alpha^{2MN-2} B(M)$  and therefore  $B \sim M^{MN-1}$ .

Since  $B$  is positive one infers for  $MN > 1$  that  $V_M$  diverges for  $M \rightarrow 0$ ,  $V_M(M \rightarrow 0) \rightarrow \infty$ . Similarly,  $V_M$  diverges logarithmically to negative values for large  $M$ . We note the special case  $M = N = 1$  with an abelian gauge symmetry. In this case  $B = \pi$  is constant and  $V_M$  can be omitted. As we have discussed in app. B this case is not a usual abelian gauge theory. Furthermore,  $B$  is invariant under  $SU(M) \times U(1)$  flavor transformations  $M \rightarrow \hat{U}^\dagger M \hat{U}$ . Indeed, such transformations of  $M$  can be accompanied by a transformation  $\chi \rightarrow \chi \hat{U}$ , leaving  $B$  invariant. In consequence,  $B(M)$  can be written in terms of flavor invariants as  $\text{tr} M^2, \text{tr} M^4$  etc.. This allows one to extract the full  $M$ -dependence of  $B$  from its evaluation for diagonal matrices.

In principle, the expectation values of all observables  $O[M]$  that can be expressed in terms of  $M$  can be computed by inserting  $O[M]$  in the functional integral (82). This includes generalized plaquette observables which consist of an ordered matrix product of links along a closed loop. Such observables can be expressed by an ordered matrix product of factors  $M$  at each point of the loop. Important properties of gauge theories can be investigated in a formulation employing only gauge singlets where the gauge transformations are no longer visible at all! Besides the formulation of scalar lattice gauge theories in terms of complex scalar fields  $\chi$ , and the equivalent formulation with additional link variables  $L$ , one therefore can employ a third equivalent “singlet formulation” in terms of a functional integral involving only the gauge singlets  $M$ . No trace of the gauge transformations is directly visible anymore - the only memory is the additional potential  $V_M$ . Since the matrices  $M$  transform non-trivially under the global flavor symmetry  $SU(M) \times U(1)$  we may also call the functional integral in terms of  $M$  the “flavor formulation”. In a wider sense, this corresponds to a formulation of gauge theories in terms of generalized “glue balls”.

For  $MN > 1$  the presence of the potential  $V_M$  in the microscopic action for  $M$

$$\begin{aligned} S_M[M] &= S_\chi[M] + S_V[M], \\ S_V[M] &= \sum_x V_M(M(x)) \end{aligned} \quad (84)$$

has important consequence. The combined potential  $\hat{V}(M) = 2d\mu^2 \text{tr} M^2 + \bar{V}(M) + V_M(M)$  has its minimum for  $M_0 \neq 0$ , provided  $\hat{V}$  diverges to positive values for large  $M$ , as for  $\hat{V} = \nu^2 \text{tr} M^2, \nu^2 > 0$ . (The term  $\sim \mu^2$  arises from  $S_{W,l}$  in eqs. (59), (56).) For a suitable choice of  $\bar{V}$  (e.g.  $\bar{V} = 0$  for  $\mu^2 > 0, \nu^2 = 2d\mu^2$ ) the minimum of  $\hat{V}$  occurs for  $M_0 = m_0 \mathbb{1}$ ,  $m_0 > 0$ . For  $M = N$  we observe that  $M^{ab} = m_0 \delta^{ab}$  is realized for  $\chi_i^a = \sqrt{m_0} \hat{U}_{ia}, \hat{U}^\dagger \hat{U} = 1$ . In turn, one finds for the link bilinear  $\tilde{L}_{ij}(x; \mu) = m_0 (\hat{U}(x) \hat{U}(x + e_\mu))_{ij}$ . Up to a gauge transformation this corresponds to  $\tilde{L}_{ij} = m_0 \delta_{ij}$ . A homogeneous link field  $L_{ij} = (2/g^2)^{1/4} \delta_{ij}$  is the starting point of perturbation theory in linear lattice gauge theory with microscopic gauge coupling  $g$ .

Despite the conceptual simplicity of a formulation in terms of gauge invariant fields  $M(x)$  the specific form of the potential  $V_M(M)$ , which encodes the constraints for

$M$ , makes the gauge invariant formulation difficult to handle in practice for many purposes. For scalar lattice gauge theory with unconstrained fields  $\chi(x)$  the polynomial form of the action is simpler. In particular, the connection to link bilinears is much more direct if one uses the scalar fields  $\chi(x)$ . The constraints on  $M(x)$  are precisely what is needed in order to permit a reformulation in terms of  $\chi(x)$ .

## VII. CONCLUSIONS

We have presented a formulation of scalar lattice gauge theory. It is based in scalar fields, located on lattice sites, rather than on unitary (or orthogonal) link variables as in standard lattice gauge theories. The action and functional measure are invariant under local gauge transformations. We have found a limit for the parameters of scalar lattice gauge theory for which the long-distance behavior is the same as for standard lattice gauge theories. In particular, one expects the usual confinement property for pure non-abelian gauge theories. This shows that gauge fields can be obtained as collective excitations or composites of “fundamental” scalar fields.

Varying the parameters of scalar lattice gauge theory one expects a rich phase diagram, with confinement as well as Higgs phases. Scalar lattice gauge theory is accessible to numerical simulations by standard Monte-Carlo methods. An exploration of the phase diagram may shed new light on different universality classes for gauge theories and their mutual connection.

For establishing that the universality class of standard lattice gauge theory is realized for a region in the parameter space of scalar lattice gauge theory we have developed linear lattice gauge theory as an intermediate step. This type of model uses unconstrained linear link variables instead of the usual unitary or orthogonal link variables. It is interesting in its own right. In particular, the inverse gauge coupling is proportional to the expectation value  $l_0^2$  of the squared linear link variable. Confinement in the non-linear version of a gauge theory can be associated to the “symmetric regime” in the linear version of a gauge theory where  $l_0$  vanishes. It will be interesting to formulate the approximate continuum limit of linear lattice gauge theory.

An extension to “fundamental” fermions instead of scalars can be made along the lines discussed in this paper. In such a “fermion lattice gauge theory” Grassmann variables  $\psi_{i,\alpha}^a(x)$  and  $\bar{\psi}_{i,\alpha}^a(x)$  replace the scalar field  $\chi_i^a(x)$  and  $(\chi_i^a(x))^*$ . The index  $\alpha$  is an additional Lorentz index, e.g.  $\alpha = 1 \dots 4$  for Dirac spinors in  $d = 4$ . The functional integral (2) or (35) is now a Grassmann functional integral. The matrices  $M^{ab}(x)$  are replaced by the gauge singlets  $M_{\alpha\beta}^{ab}(x) = \bar{\psi}_{i,\alpha}^a(x) \psi_{i,\beta}^b(x)$ . In particular, the combinations  $M_{\alpha\beta}^{ab} \delta^{\alpha\beta}$  and  $M_{\alpha\beta}^{ab} \bar{\gamma}^{\alpha\beta}$  (with  $\bar{\gamma}$  the generalization of  $\gamma^5$ ) transform as scalars or pseudoscalars and can be associated with mesons. Bilinear link variables  $\tilde{L}_{ij}$  are constructed similar to eq. (8) as Lorentz and flavor singlets. One may again construct a gauge invariant action in terms of  $\psi$  and  $\bar{\psi}$  and find a region in parameter space where the universality class is the same as a gauge theory coupled to

fermions. In this way one may realize QCD with quarks in a formulation which only involves fermionic fields.

## APPENDIX A: ACTIONS WITH LOCAL GAUGE INVARIANCE

In this appendix we display details and extensions for the action of scalar lattice gauge theory. The action  $S[\chi]$  is invariant under local gauge transformations if it only involves local invariants, as

$$\begin{aligned} M^{ab}(x) &= (\chi_i^a(x))^* \chi_i^b(x), \quad I(x) = (\chi_i^a(x))^* \chi_i^a(x), \\ R^{abcd}(x) &= (\chi_i^a(x))^* (\lambda_z)_{ij} \chi_j^b(x) (\chi_k^c(x))^* (\lambda_z)_{kl} \chi_l^d(x). \end{aligned} \quad (\text{A.1})$$

Here  $\lambda_z$  are the generators of  $SU(N)$  and summation over repeated indices is implied. The precise local gauge symmetry depends on which local invariants appear in the action. (Note  $I = \text{tr} M$  if  $M^{ab}$  is interpreted as an  $M \times M$  matrix.) If  $S$  involves only  $M^{ab}$  and  $R^{abcd}$  the local gauge symmetry is  $SU(N) \times U(1)$ . We may also consider real scalar fields -  $R^{abcd}$  is not defined in this case. If  $S$  can be written in terms of the real bilinears  $M^{ab}(x)$  the local gauge symmetry is  $SO(N)$ .

Invariants that are singlets with respect to  $SU(N)$ , but charged with respect to the  $U(1)$ -subgroup, involve  $N$  in powers of  $\chi$  (for  $M \geq N$ )

$$D^{a_1 a_2 \dots a_N}(x) = \epsilon^{i_1 i_2 \dots i_N} \chi_{i_1}^{a_1}(x) \chi_{i_2}^{a_2}(x) \dots \chi_{i_N}^{a_N}(x). \quad (\text{A.2})$$

Models with action formulated in terms of  $M, R$  and  $D$  are invariant under local  $SU(N)$ -gauge transformations. The abelian  $U(1)$  symmetry becomes a global symmetry if all terms in the action involve an equal number of  $D$  and  $D^*$ . However, there will be no local  $U(1)$ -symmetry if  $S$  contains derivatives of  $D$ , such that the local gauge group is  $SU(N)$ . This will be a way to construct pure QCD based on the gauge group  $SU(3)$ . In addition to the local gauge symmetry one may have additional global symmetries acting on the flavor indices.

We concentrate on models with local  $SU(N) \times U(1)$  gauge symmetry. Let us consider different types of terms that may appear in the action. A potential term is given by

$$S_V = \sum_x V(M^{ab}(x), R^{abcd}(x), D^*(x)D(x)). \quad (\text{A.3})$$

In the following we will not use  $R$ , nor other possible higher order invariants as  $D^*D$ . They can be added if needed. If the action contains only  $S_V$  the model is a simple ultra-local theory where different lattice points are independent of each other.

We next introduce

$$K_\mu^{ab}(x) = (\chi_i^a(x+e_\mu))^* \chi_i^b(x+e_\mu) - (\chi_i^a(x))^* \chi_i^b(x), \quad (\text{A.4})$$

with  $e_\mu$  a unit vector in one of the  $d$  lattice directions. A term

$$S_K = \sum_x (K_\mu^{ab}(x))^* K_\mu^{ab}(x) \quad (\text{A.5})$$

links different lattice sites such that the partition function is no longer a trivial product of individual site contributions. It is invariant under global (not local!)  $SU(M)$  flavor transformations, as well as under lattice rotations by  $\pi/2$ . (We discuss here euclidean lattice gauge theories, with a possible generalization to Minkowski signature or, alternatively, analytic continuation of the correlation functions to Minkowski space.)

Using the lattice derivatives (4) one sees that  $S_K$  amounts to a kinetic term for the invariant  $M^{ab}$

$$S_K = a^2 \sum_x \partial_\mu (M^{ab}(x))^* \partial_\mu M^{ab}(x). \quad (\text{A.6})$$

This can be generalized to other terms involving derivatives of the local invariants  $M$  or  $R$ . We can express  $S_K$  in terms of the link bilinears (8), using

$$\begin{aligned} (K_\mu^{ab}(x))^* K_\mu^{ab}(x) &= (M^{ab}(x+e_\mu))^* M^{ab}(x+e_\mu) \\ &+ (M^{ab}(x))^* M^{ab}(x) - 2\tilde{L}_{ij}(x; \mu) \tilde{L}_{ij}^*(x; \mu), \end{aligned} \quad (\text{A.7})$$

(no summation over  $\mu$  here). This yields  $S_K$  as a sum over links denoted by  $(x; \mu)$ , plus a potential term,

$$\begin{aligned} S_K &= 2d \sum_x (M^{ba}(x))^\dagger M^{ab}(x) \\ &- 2 \sum_{\text{links}} \tilde{L}_{ji}^\dagger(x; \mu) \tilde{L}_{ij}(x; \mu). \end{aligned} \quad (\text{A.8})$$

Here the sum over links amounts to  $\sum_x \sum_\mu$ .

We have already introduced in sect. II the plaquette invariants  $\tilde{P}(x; \mu)$ , cf. eq. (10). They correspond to a product of links around a plaquette. A further type of invariants,

$$\begin{aligned} \tilde{Q}(x; \mu, \nu) &= \tilde{Q}(x; \nu, \mu) = \tilde{Q}^*(x; \mu, \nu) \\ &= \text{tr} \{ \tilde{L}(x; \mu) \tilde{L}^\dagger(x; \mu) \tilde{L}(x; \nu) \tilde{L}^\dagger(x; \nu) \}, \end{aligned} \quad (\text{A.9})$$

corresponds to a sequence of links from  $x$  to  $x+e_\mu$  and back, and then to  $x+e_\nu$  and back. The combination

$$\begin{aligned} \mathcal{L}_p(x; \mu, \nu) &= -\frac{1}{2} [\tilde{P}(x; \mu, \nu) + \tilde{P}(x; \nu, \mu)] \\ &+ \frac{1}{4} [\tilde{Q}(x; \mu, \nu) + \tilde{Q}(x+e_\mu, \nu, -\mu) \\ &+ \tilde{Q}(x+e_\mu+e_\nu, -\mu, -\nu) + \tilde{Q}(x+e_\nu, -\nu, \mu)] \end{aligned} \quad (\text{A.10})$$

can be viewed as a type of locally gauge invariant kinetic term for the collective link variables. It can be written in the form (15), such that  $\mathcal{L}_p \geq 0$ .

We can express  $\tilde{Q}$  in terms of  $M$  as

$$\tilde{Q}(x; \mu, \nu) = \text{tr} \{ M(x) M(x+e_\mu) M(x) M(x+e_\nu) \}. \quad (\text{A.11})$$

In terms of the scalar invariants  $M$  one finds

$$\begin{aligned} \mathcal{L}_p &= \frac{1}{4} \text{tr} \{ M(x+e_\nu) [M(x+e_\mu+e_\nu) - M(x)] \\ &\times M(x+e_\mu) [M(x+e_\mu+e_\nu) - M(x)] \\ &+ M(x) [M(x+e_\mu) - M(x+e_\nu)] M(x+e_\mu+e_\nu) \\ &\times [M(x+e_\mu) - M(x+e_\nu)] \}. \end{aligned} \quad (\text{A.12})$$

This coincides with eq. (17).

In this paper we investigate models which involve  $\mathcal{L}_p$  with a negative sign. In order to have a plaquette action that is bounded from below we have to combine  $-\mathcal{L}_p$  with some other piece,  $\mathcal{S}_p = \mathcal{A}_p - \mathcal{L}_p \geq 0$ . With

$$\begin{aligned} A_{\pm} &= M(x + e_{\mu}) \pm M(x + e_{\nu}), \\ B_{\pm} &= M(x + e_{\mu} + e_{\nu}) \pm M(x), \end{aligned} \quad (\text{A.13})$$

one has

$$\begin{aligned} \mathcal{L}_p &= \frac{1}{16} \text{tr} \{ A_+ B_- A_+ B_- + B_+ A_- B_+ A_- \\ &\quad - 2 A_- B_- A_- B_- \}. \end{aligned} \quad (\text{A.14})$$

We define

$$\mathcal{A}_p = \frac{1}{16} \text{tr} \{ A_+^2 B_-^2 + B_+^2 A_-^2 + 2 A_-^2 B_-^2 \} \geq 0. \quad (\text{A.15})$$

For hermitean matrices  $K, L$  one has the identities

$$\begin{aligned} \text{tr} \{ K^2 L^2 - K L K L \} &= \frac{1}{2} \text{tr} \{ [K, L]^{\dagger} [K, L] \}, \\ \text{tr} \{ K^2 L^2 + K L K L \} &= \frac{1}{2} \text{tr} \{ \{K, L\}^{\dagger} \{K, L\} \}, \\ \text{tr} \{ K^2 L^2 \} &= \frac{1}{4} \text{tr} \{ \{K, L\}^{\dagger} \{K, L\} \} \\ &\quad + \frac{1}{4} \text{tr} \{ [K, L]^{\dagger} [K, L] \}. \end{aligned} \quad (\text{A.16})$$

This shows that the combination

$$\begin{aligned} \mathcal{S}_p &= \mathcal{A}_p - \mathcal{L}_p = \frac{1}{16} \text{tr} \{ A_+^2 B_-^2 - A_+ B_- A_+ B_- \\ &\quad + B_+^2 A_-^2 - B_+ A_- B_+ A_- \\ &\quad + 2 A_-^2 B_-^2 + 2 A_- B_- A_- B_- \} \geq 0 \end{aligned} \quad (\text{A.17})$$

is positive semidefinite. It coincides with eq. (7).

One may express  $\mathcal{L}_p$  and  $\mathcal{A}_p$  in terms of lattice derivatives of  $M$ . For this purpose we note that  $A_-$  and  $B_-$  are derivatives,

$$\begin{aligned} M(x + e_{\mu}) - M(x + e_{\nu}) &= a \{ \partial_{\mu} M(x) - \partial_{\nu} M(x) \}, \\ M(x + e_{\mu} + e_{\nu}) - M(x) &= \frac{a}{2} \{ \partial_{\mu} [M(x + e_{\nu}) + M(x)] \\ &\quad + \partial_{\nu} [M(x + e_{\mu}) + M(x)] \}. \end{aligned} \quad (\text{A.18})$$

We may also employ

$$\begin{aligned} \partial_{\mu} \partial_{\nu} M(x) &= \partial_{\nu} \partial_{\mu} M(x) = \frac{1}{a} \partial_{\nu} [M(x + e_{\mu}) - M(x)] \\ &= \frac{1}{a^2} [M(x + e_{\mu} + e_{\nu}) - M(x + e_{\mu}) \\ &\quad - M(x + e_{\nu}) + M(x)] \end{aligned} \quad (\text{A.19})$$

in order to express  $\mathcal{L}_p$  in terms of  $M(x), \partial_{\mu} M(x), \partial_{\nu} M(x)$  and  $\partial_{\mu} \partial_{\nu} M(x)$ .

The leading term in the continuum limit reads

$$\begin{aligned} \mathcal{L}_p(x; \mu, \nu) &= \frac{a^2}{2} \text{tr} \{ M \partial_{\mu} M M \partial_{\mu} M + M \partial_{\nu} M M \partial_{\nu} M \} \\ &\quad + O(a^4). \end{aligned} \quad (\text{A.20})$$

With  $a^d \sum_x = \int d^d x = \int_x$  and an appropriate rescaling of  $M, M \rightarrow a^{(d-2)/4} M$  this results in

$$S_p = \int_x \frac{d-1}{2} \text{tr} \{ M \partial_{\mu} M M \partial_{\mu} M \}. \quad (\text{A.21})$$

For  $d = 4$  we recognize the last term in eq. (1) as  $-S_p$ . Similarly, one finds in leading order

$$\begin{aligned} \mathcal{A}_p(x; \mu, \nu) &= \frac{a^2}{2} \text{tr} \{ M^2 (\partial_{\mu} M \partial_{\mu} M + \partial_{\nu} M \partial_{\nu} M) \} \\ &\quad + O(a^4). \end{aligned} \quad (\text{A.22})$$

This can be combined with the continuum limit of the link action

$$\begin{aligned} \mathcal{S}_l &= a^2 \left( \bar{\lambda} - \frac{d-1}{2} \right) \text{tr} \{ M^2 (\partial_{\mu} M)^2 \} \\ &\quad + \frac{\bar{m}^2}{d} a^{\frac{d+2}{2}} \text{tr} M^2. \end{aligned} \quad (\text{A.23})$$

For  $d = 4$  this yields the naive continuum action (1). We emphasize, however, that the detailed lattice action is necessary for an understanding of our model. For example the crucial importance of the plaquette invariant  $\tilde{P}$  is not visible in the naive continuum formulation.

Finally, we observe that the rescaled continuum field  $M_c(x) = M(x) a^{(2-d)/4}$  has dimension  $\text{mass}^{(d-2)/4}$ . For  $d = 4$  it scales  $\sim \sqrt{\text{mass}}$ , different from the more familiar scaling of scalars  $\sim \text{mass}$ . This is due to the absence of a quadratic kinetic term, while derivative terms involve four powers of  $M$ . Correspondingly,  $\bar{m}^2$  has dimension  $\text{mass}^3$  and  $\bar{\lambda}$  is dimensionless. The scalar field  $\chi$  scales with  $(\text{mass})^{1/4}$ .

## APPENDIX B: PHASE FLUCTUATIONS AND CRITICAL FLAVOR NUMBER

The dynamics of lattice gauge theories is closely related to phase fluctuations. Indeed, the standard lattice gauge theories are formulated in terms of link variables that are unitary matrices. These unitary matrices can be considered as generalized phases. In scalar lattice gauge theories the issue of phases depends on the number of flavors  $M$ . This will impose restrictions on  $M$  if we want to obtain the same universality class as standard lattice gauge theories.

As a first example we consider the case  $N = 1$  with abelian  $U(1)$ -gauge symmetry. For  $M = 1$  the scalar  $\chi$  is a single complex field that we may write as  $r(x) \exp[i\alpha(x)]$ , with real  $r \geq 0$ . The definitions (3), (8) imply

$$\begin{aligned} M(x) &= r^2(x), \\ \tilde{L}(x; \mu) &= r(x) r(x + e_{\mu}) e^{i(\alpha(x) - \alpha(x + e_{\mu}))}. \end{aligned} \quad (\text{B.1})$$

It is obvious that the phases  $\alpha(x)$  do not appear in the action - they are pure gauge degrees of freedom. We can define the phase of the link bilinear  $\tilde{L}(x; \mu)$  by  $\beta(x; \mu) = \alpha(x) - \alpha(x + e_{\mu})$ . The sum of phases  $\beta$  for a product of

link bilinears around a plaquette is constrained to vanish. Such a model is not expected to share similar properties as a standard abelian gauge theory.

The situation is different for  $M = 2$ . We have now two complex scalar fields  $\chi^a(x)$ , and obtain with the two phases  $\alpha_{1,2}(x)$

$$\begin{aligned}\chi^a(x) &= r_a(x) \exp[i\alpha_a(x)], \\ M^{ab}(x) &= r_a(x)r_b(x) \exp[i(\alpha_b(x) - \alpha_a(x))], \\ \tilde{L}(x; \mu) &= r_1(x)r_1(x + e_\mu) \exp[i(\alpha_1(x) - \alpha_1(x + e_\mu))] \\ &\quad + r_2(x)r_2(x + e_\mu) \exp[i(\alpha_2(x) - \alpha_2(x + e_\mu))].\end{aligned}\quad (\text{B.2})$$

A non-trivial phase  $\alpha_2 - \alpha_1$  is present in  $M$ . For the phase  $\beta(x; \mu)$  of the link variable one has

$$\begin{aligned}tg\beta(x; \mu) & \\ &= \frac{\sum_a r_a(x)r_a(x + e_\mu) \sin[\alpha_a(x) - \alpha_a(x + e_\mu)]}{\sum_a r_a(x)r_a(x + e_\mu) \cos[\alpha_a(x) - \alpha_a(x + e_\mu)]}.\end{aligned}\quad (\text{B.3})$$

There is no constraint anymore that the sum of  $\beta$  around a plaquette has to vanish. If the phase fluctuations play a decisive role such a model may be expected to belong to the same universality class as a standard abelian lattice gauge theory.

These findings can be generalized to arbitrary  $N$ . For any given  $N$  we expect that there is a critical flavor number  $M_c(N)$  such that for  $M \leq M_c$  the phase fluctuations are too much constrained such that scalar lattice gauge theory cannot belong to the standard universality class of gauge theories. We have seen already that this is the case for  $M = 1$  for arbitrary  $N$ , such that  $M_c(N) \geq 1$ . In the opposite, for  $M > M_c(N)$  there exists a region in the parameter space of scalar lattice gauge theory for which the universality class of a standard gauge theory is realized.

An estimate of  $M_c(N)$  is a difficult task. In order to get some intuition, we present here some simple counts of degrees of freedom. First, we observe that for a periodic lattice with  $\mathcal{N}$  sites the total number of real degrees of freedom contained in  $\chi$  is  $2NM\mathcal{N}$ . On the other hand, for unconstrained link variables the total number of real degrees of freedom is  $2N^2d\mathcal{N}$ . For  $M < dN$  the link bilinears  $\tilde{L}(x; \mu)$  defined by eq. (14) are not all independent but rather have to obey constraints. One may speculate that for  $M \geq dN$  (and  $d > 1$ ) the phase fluctuations are

sufficiently unconstrained in order to admit the universality class of standard gauge theories. This would imply  $M_c(N) < dN$ .

A second counting concerns the degrees of freedom appearing in a simple plaquette term  $\mathcal{L}_p$ . There are four sites and therefore  $8NM$  real degrees of freedom in the variables  $\chi(x)$ . Due to the local gauge symmetry not all of them appear in  $\mathcal{L}_p$ . An upper bound for the number of independent degrees of freedom appearing in  $\mathcal{L}_p$  is  $4N(2M - N)$ . (The number of real degrees of freedom in unconstrained hermitean fields  $M(x)$  is  $4M^2$ . For  $M > N$  the fields  $M(x)$  obey constraints beyond hermiticity.) For unconstrained link variables we can use the gauge transformations in order to bring three link variables in the plaquette to a hermitean form, accounting for  $3N^2$  degrees of freedom. This is not possible for the fourth link variable which retains the  $2N^2$  degrees of freedom of an arbitrary complex matrix. The total number of link degrees of freedom appearing in  $\mathcal{L}_p$  amounts therefore to  $5N^2$ . For  $5N > 8M - 4N$  the links within each plaquette must obey local constraints. If these local constraints are strong enough to forbid the realization of the standard universality class one would infer  $M_c(N) \geq (9/8)N$ .

An interesting case is  $M = N$  for which  $\chi, M$  and  $\tilde{L}$  are all  $N \times N$  matrices. We can write an arbitrary complex matrix  $\chi$  in the form

$$\chi(x) = V(x)D(x)\tilde{V}^\dagger(x) \quad (\text{B.4})$$

with real diagonal matrix  $D(x)$  and unitary matrices  $V(x)$  and  $\tilde{V}(x)$ . This implies

$$\begin{aligned}M(x) &= \tilde{V}(x)D^2(x)\tilde{V}^\dagger(x), \\ \tilde{L}(x; \mu) &= V(x)D(x)\tilde{V}^\dagger(x)\tilde{V}(x + e_\mu)D(x + e_\mu)V^\dagger(x + e_\mu).\end{aligned}\quad (\text{B.5})$$

We can associate  $V(x)$  with the gauge degrees of freedom that do not appear in the action. (One may set  $V(x) = 1$ .) The matrix  $M(x)$  has positive eigenvalues given by  $D^2(x)$ . For  $D^2(x)$  proportional to the unit matrix,  $D^2(x) = m(x)$ , one finds  $M(x) = m(x)$ . In this case the phases contained in  $\tilde{V}(x)$  do not appear in the action. For different eigenvalues of  $M(x)$ , however, the action will depend on the phases  $\tilde{V}(x)$ . It is not clear if the standard universality class can be realized for  $M = N$  or not.

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